

NEW Fixed Point Results In I-Generalized Metric Space By Generalized Geraghty Contraction

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Abstract-This work aims to further generalization of generalized $\alpha - \psi$ -Geraghty contraction and establish a new fixed point result in I-g.m.s. and its corresponding version in g.m.s. Also, further generalization of generalized \mathcal{L} -contraction and establish a new fixed point theorem in I-g.m.s. its corresponding version in g.m.s.

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1. Introduction

To form models of several real life problems mathematically, fixed point theory plays an important role, which created its own glorious place in mathematics in the last 50 years. Geraghty [6] introduced auxiliary functions and refined Banach contraction principle [3] in metric space in 1973, stated as:

“Let (X, d) be a complete metric space and $T: X \rightarrow X$, $\beta: [0, \infty) \rightarrow [0, 1)$ satisfy

(i) $\beta(w_n) \rightarrow 1 \implies w_n \rightarrow 0$

(ii) $d(Tx, Ty) \leq \beta(d(x, y))d(x, y)$ for all $x, y \in X$.

Then T has a unique fixed point.”

Branchiari introduced g.m.s. in 2000 [4] and from then several fixed point results have been developed in this space.

In 2012, Samet et. al. [9] introduced $\alpha - \psi$ contractive map and proved some generalized fixed point results in metric spaces.

In 2014, Asadi et. al. [2] introduced generalized $\alpha - \psi$ -Geraghty contractive map and established fixed point results in g.m.s.

In 2019, Suman and Biswas introduced [10] I-g.m.s. and proved some general fixed point results there.

Here we shall further generalize this generalized α - ψ -Geraghty contraction, called, IJ-g α - ψ -Geraghty contraction in I-g.m.s. and g.m.s., and establish a new fixed point theorem in I-g.m.s., and then get the corresponding version of this fixed point theorem in g.m.s. by means of a corollary.

Jleli and Samet [7] introduced Θ -contraction in g.m.s. in 2014, stated as:

“Let (X, d) be a g.m.s. A map $T: X \rightarrow X$ is said to be Θ -contraction if $\exists \theta \in \Omega_{1,2,3}, k \in (0,1)$ such that $\forall x, y \in X$,

$d(Tx, Ty) > 0 \implies \theta(d(Tx, Ty)) \leq [\theta(d(Tx, Ty))]^k$, where $\Omega_{1,2,3}$ is the class of the functions

$\theta: (0, \infty) \rightarrow (1, \infty)$ satisfying

(Θ_1) θ is nondecreasing.

(Θ_2) For each sequence $\{t_n\}$ in $(0, \infty)$, $\lim_{n \rightarrow \infty} \theta(t_n) = 1$ iff $\lim_{n \rightarrow \infty} t_n = 0$.

(Θ_3) $\exists r \in (0,1), l \in (0, \infty]$ such that $\lim_{t \rightarrow 0^+} \frac{\theta(t)-1}{t^r} = l$.”

Ahmad et al. [1] replaced the condition (Θ_3) by “(Θ_4) θ is continuous.” $\Omega_{1,2,4}$ denotes the class of functions satisfying (Θ_1), (Θ_2), (Θ_4).

Cho [5] introduced \mathcal{L} -contraction which unify many contractions including Θ -contractions in g.m.s. in 2018 stated as:

“Let (X, d) be a g.m.s. A map $T: X \rightarrow X$ is said to be an \mathcal{L} -contraction with respect to $\zeta \in \mathcal{L}$ if there exists $\theta \in \Omega_{1,2,4}$ such that $\forall x, y \in X, d(Tx, Ty) > 0 \implies \zeta[\theta(d(Tx, Ty)), \theta(d(x, y))] \geq 1$, where \mathcal{L} is the class of functions $\zeta: [1, \infty)^2 \rightarrow \mathbb{R}$ satisfying

(\mathcal{L}_1) $\zeta(1,1) = 1$, (\mathcal{L}_2) $\zeta(t, s) < \frac{s}{t}, \forall t, s > 1$,

(\mathcal{L}_3) If $\{t_n\}$ and $\{s_n\}$ are two sequences in $(1, \infty)$ with $t_n < s_n$ such that $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n > 1$ then $\overline{\lim}_{n \rightarrow \infty} \zeta(t_n, s_n) < 1$.”

In 2020, Saleh H.N. et al. [8] introduced generalized \mathcal{L} -contractions and proved fixed-point theorem in g.m.s.

Here we shall further generalize this generalized \mathcal{L} -contraction, called IJg- \mathcal{L} -contraction in I-g.m.s. and g.m.s., and establish a new fixed point theorem in I-g.m.s. Then we shall get the corresponding version of this result in g.m.s. by means of a corollary.

2. Preliminaries

We recall some notations and definitions.

“Definition(2.1) [Geraghty function] [6] A function $\beta : [0, \infty) \rightarrow [0,1)$ satisfying $\lim_{n \rightarrow \infty} \beta(t_n) = 1 \implies \lim_{n \rightarrow \infty} t_n = 0$, for all sequences $\{t_n\}$ of nonnegative real numbers, is called a Geraghty function. The class of all Geraghty functions is denoted by \mathcal{F} .”

“Definition(2.2) [ψ -function] A function $\psi : [0, \infty) \rightarrow [0, \infty)$ is called a ψ -function if ψ is nondecreasing, continuous and $\forall t \in [0, \infty), \psi(t) = 0$ iff $t = 0$. The class of all ψ -functions is denoted by Ψ .”

“Definition(2.3) [Triangular α -admissible function] [2] Let X be a nonempty set, $T : X \rightarrow X$ and $\alpha : X^2 \rightarrow \mathbb{R}$. Then T is called triangular α -admissible if
 (C1) $x, y \in X, \alpha(x, y) \geq 1 \implies \alpha(Tx, Ty) \geq 1$,
 (C2) $x, y, z \in X, \alpha(x, z) \geq 1, \alpha(z, y) \geq 1 \implies \alpha(x, y) \geq 1$.”

“Definition(2.4) [g.m.s.] [4] For a nonempty set X and a map $d : X^2 \rightarrow [0, \infty)$, the order pair (X, d) is said to be a generalized metric space (g.m.s. in short) if $\forall x, y, u, v \in X$ with $u \neq v$, and u, v are distinct from x, y ,

- (gm1) $d(x, y) = 0 \iff x = y$
- (gm2) $d(x, y) = d(y, x)$
- (gm3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.”

“Definition(2.5) [generalized α - ψ -Geraghty contraction][2] Let (X, d) be a g.m.s. and $\alpha : X^2 \rightarrow \mathbb{R}$. A map $T : X \rightarrow X$ is called generalized α - ψ -Geraghty contraction if $\exists \beta \in \mathcal{F}$ such that $\forall x, y \in X$,
 $\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y))$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty)\}$ and $\psi \in \Psi$.”

“Definition(2.6) [I-uniqueness] [10] Let X be a non-empty set and $f : X \rightarrow X$ be an idempotent map. Two elements x, y of X are said to be I-unique with respect to f , or simply I-unique if $fx = fy$; otherwise x and y are said to be I-distinct elements of X .”

“Definition(2.7) [I-g.m.s.] [10] Let X be a non-empty set and $f : X \rightarrow X$ be an idempotent map. A map $d : X^2 \rightarrow [0, \infty)$ is said to be an I-generalized metric on X if $\forall x, y, u, v \in X$ with $fu \neq fv$, and u, v are I-distinct from x, y ,

- (Igm1) $d(x, fy) = 0 \iff fx = fy$ and $d(fx, y) = 0 \iff fx = fy$.
- (Igm2) $d(x, fy) = d(y, fx)$ and $d(fx, y) = d(fy, x)$
- (Igm3) $d(x, y) \leq d(fx, u) + d(fu, v) + d(v, fy)$.

The order triple (X, d, f) is called I-generalized metric space (I-g.m.s. in short).”

“Example(2.8) [10] Let $X = A \cup B$, where $A = \{0,2\}$,

$B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Let $f : X \rightarrow X$ be an idempotent map. Define $d : X^2 \rightarrow [0, \infty)$ by $\forall x, y \in X, d(x, y) =$

$$\begin{cases} 0, & \text{if } fx = fy \\ 1, & \text{if } fx \neq fy \text{ and } \{fx, fy\} \subset A \text{ or } \{fx, fy\} \subset B \\ fy, & \text{if } fx \in A, fy \in B \\ fx, & \text{if } fx \in B, fy \in A \end{cases}$$

Then (X, d, f) is an I-g.m.s.”

“Theorem(2.9) [10] Let (X, d, f) be an I-g.m.s. Then

- (i) $d(x, x) = 0, \forall x \in X.$
- (ii) $d(x, fy) = d(fx, y) = d(fx, fy) = d(fy, fx) \geq d(x, y), d(y, x), \forall x, y \in X.$
- (iii) $d(x, fx) = 0, \forall x \in X.”$

“Definition(2.10) [10] A sequence $\{x_n\}$ in an I-g.m.s. (X, d, f) is said to “I-converge” to a point x of X , if for every $\epsilon > 0, \exists m \in \mathbb{N}$ such that $d(fx_n, x) < \epsilon, \forall n \geq m.$
 A sequence which is not I-convergent in an I-g.m.s. (X, d, f) is called a non I-convergent or an I-divergent sequence.”

“Definition(2.11) [10] A sequence $\{x_n\}$ in an I-g.m.s. (X, d, f) is said to be an I-cauchy sequence in X if corresponding to every $\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that $d(fx_m, x_n) < \epsilon, \forall m, n \geq n_0,$ i.e., $d(fx_{n+p}, x_n) < \epsilon, \forall n \geq n_0, \forall p \geq 1.”$

“Definition(2.12) [10] An I-g.m.s. (X, d, f) is said to be I-complete if every I-cauchy sequence in X I-converges to some point of X ; otherwise (X, d, f) is called I-incomplete.”

“Definition(2.13) [10] Let X be a nonempty set and $f : X \rightarrow X$ is an idempotent map. A map $h : X \rightarrow X$ is said to have an I-fixed point x in X if $(fh)x = fx.”$

“Definition(2.14) [generalized \mathcal{L} -contraction] [8] Let (X, d) be a Branciari metric space and $T : X \rightarrow X.$ Then T is said to be a generalized \mathcal{L} -contraction with respect to $\zeta \in \mathcal{L}$ if there exists $\theta \in \Omega_{1,2,4}$ and a constant $L \geq 0$ such that for all $x, y \in X, d(Tx, Ty) > 0 \implies \zeta \left(\theta(d(Tx, Ty)), \theta(d(x, y) + LN(x, y)) \right) \geq 1,$ where $N(x, y) = \min \left\{ d(x, Ty), d(y, Tx), \frac{d(x, Tx) + d(y, Ty)}{2} \right\}.”$

“Theorem(2.15) [8] Let (X, d) be a complete g.m.s. and $T : X \rightarrow X.$ If T is a continuous generalized \mathcal{L} -contraction map with respect to $\zeta \in \mathcal{L},$ then it has a unique fixed point.”

3. Main results

Let us start with following definition.

Definition(3.1) [IJ-g α - ψ -Geraghty contraction] Let (X, d, f) be an I-g.m.s., and $\alpha : X^2 \rightarrow \mathbb{R}$. A map $T : X \rightarrow X$ is called IJ-g α - ψ -Geraghty contraction, if there exists a Geraghty function β such that $\forall x, y \in X$,

$$\alpha(fx, y)\psi(d((fT)x, Ty)) \leq \beta(\psi_1(M(fx, y)))\psi_2(M(fx, y)), \text{ where } \psi, \psi_1, \psi_2 \in \Psi \text{ and } M(x, y) = \max\{d(fx, y), d(fx, Tx), d(fy, Ty), d(fy, Tx), d(fy, T^2x)\}$$

Replacing the idempotent map f by the identity map I_X on X we shall get more generalization of generalized α - ψ -Geraghty contraction in g.m.s., called J-g α - ψ -Geraghty contraction.

Definition(3.2) [Triangular I- α -admissible function] Let (X, d, f) be an I-g.m.s., $T : X \rightarrow X$ and $\alpha : X^2 \rightarrow \mathbb{R}$. Then T is called triangular I- α -admissible if

- (a) $x, y \in X, \alpha(fx, y) \geq 1 \Rightarrow \alpha((fT)x, Ty) \geq 1$.
- (b) $x, y, z \in X, \alpha(fx, z) \geq 1, \alpha(fy, z) \geq 1 \Rightarrow \alpha(fx, y) \geq 1$.

Theorem(3.3) Let (X, d, f) be an I-complete I-g.m.s., $\alpha : X^2 \rightarrow \mathbb{R}$ and $T : X \rightarrow X$ satisfy

- (i) T is an IJ-g α - ψ -Geraghty contraction map such that $\psi_2(t) \leq \psi(t), \forall t > 0$.
- (ii) T is triangular I- α -admissible.
- (iii) There exists $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$ and $\alpha(fx_0, T^2x_0) \geq 1$.
- (iv) T is I-continuous.

Then T has an I-fixed point u in X and $\{T^n x_0\}$ I-converges to u .

Proof: Let $x_0 \in X$ such that $\alpha(fx_0, Tx_0) \geq 1$ and $\alpha(fx_0, T^2x_0) \geq 1$ (By (iii)). Let $x_k = Tx_{k-1}, \forall k \in \mathbb{N}$. If $fx_k = fx_{k-1}$ for some $k \in \mathbb{N}$, then x_{k-1} is an I-fixed point of T . Let $fx_k \neq fx_{k-1}, \forall k \in \mathbb{N}$.

T is triangular I- α -admissible implies that $\alpha(fx_1, x_2) = \alpha((fT)x_0, Tx_1) \geq 1$ and $\alpha(fx_1, x_3) = \alpha((fT)x_0, T^2x_1) \geq 1$

(By (iii)).

Let $\alpha(fx_k, x_{k+1}) \geq 1$ and $\alpha(fx_k, x_{k+2}) \geq 1$.

Now $\alpha(fx_{k+1}, x_{k+2}) = \alpha((fT)x_k, Tx_{k+1}) \geq 1$ and $\alpha(fx_{k+1}, x_{k+3}) = \alpha((fT)x_k, T^2x_{k+1}) \geq 1$ (By (iii)).

Therefore by mathematical induction, $\alpha(fx_k, x_{k+1}) \geq 1$ and $\alpha(fx_k, x_{k+2}) \geq 1, \forall k \in \mathbb{N}$.

(1)

Since T is triangular I- α -admissible, from (1) we get $\alpha(fx_{k+2}, x_{k+3}) \geq 1$ and $\alpha(fx_{k+2}, x_{k+4}) \geq 1, \forall k \in \mathbb{N}$.

(2)

Since T is triangular I- α -admissible, from (1) and (2) we get $\alpha(fx_k, x_{k+3}) \geq 1$.

By induction, we shall get $\alpha(fx_k, x_{k+p}) \geq 1, \forall k, p \in \mathbb{N}$.

(3)

$$\text{Now by (i) we get } \psi(d(fx_k, x_{k+1})) \leq \alpha(fx_{k-1}, x_k)\psi(d((fT)x_{k-1}, Tx_k)) \leq \beta(\psi_1(M(fx_{k-1}, x_k)))\psi_2(M(fx_{k-1}, x_k))$$

$$< \psi_2(M(fx_{k-1}, x_k))$$

(4)

where $M(fx_{k-1}, x_k) =$

$$\max\{\max\{d(fx_{k-1}, x_k), d(fx_{k-1}, x_k), d(fx_k, x_{k+1}), d(fx_k, x_k), d(fx_k, x_{k+1})\}\}$$

$$= \max\{d(fx_{k-1}, x_k), d(fx_k, x_{k+1})\}.$$

(5)

If $M(fx_{k-1}, x_k) = d(fx_k, x_{k+1})$, then from (4) we get

$$\psi(d(fx_k, x_{k+1})) < \psi_2(d(fx_k, x_{k+1})) \leq \psi(d(fx_k, x_{k+1})) \text{ (By (i)), a contradiction.}$$

Therefore $M(fx_{k-1}, x_k) = d(fx_{k-1}, x_k)$.

(6)

Now (6) implies that $\psi(d(fx_k, x_{k+1})) < \psi_2(d(fx_{k-1}, x_k))$ (By (4)) $\leq \psi(d(fx_{k-1}, x_k))$ (By (i)).

$$\Rightarrow d(fx_k, x_{k+1}) < d(fx_{k-1}, x_k), \forall k \in \mathbb{N} \text{ (since } \psi \in \Psi)$$

(7)

Therefore the sequence $\{d(fx_k, x_{k+1})\}$ is a decreasing sequence of nonnegative real numbers, so that it converges to some $r \in [0, \infty)$. Let $r > 0$. Then from (4) we get

$$\frac{\psi(d(fx_k, x_{k+1}))}{\psi_2(M(fx_{k-1}, x_k))} \leq \beta(\psi_1(M(fx_{k-1}, x_k))) < 1$$

$$\Rightarrow \frac{\psi(d(fx_k, x_{k+1}))}{\psi_2(d(fx_{k-1}, x_k))} \leq \beta(\psi_1(d(fx_{k-1}, x_k))) < 1$$

$$\Rightarrow \frac{\psi(d(fx_k, x_{k+1}))}{\psi(d(fx_{k-1}, x_k))} \leq \beta(\psi_1(d(fx_{k-1}, x_k))) < 1 \text{ (since } \psi_2(t) \leq \psi(t), \forall t > 0).$$

$$\Rightarrow \lim_{k \rightarrow \infty} \beta(\psi_1(d(fx_{k-1}, x_k))) = 1 \text{ (By continuity of } \psi)$$

$$\Rightarrow \lim_{k \rightarrow \infty} \psi_1(d(fx_{k-1}, x_k)) = 0 \text{ (By property of } \beta).$$

$$\Rightarrow \psi_1(\lim_{k \rightarrow \infty} d(fx_{k-1}, x_k)) = 0 \text{ (By continuity of } \psi_1).$$

$$\Rightarrow \lim_{k \rightarrow \infty} d(fx_{k-1}, x_k) = 0 \text{ (By property of } \psi_1).$$

$$\Rightarrow \lim_{k \rightarrow \infty} d(fx_k, x_{k+1}) = 0. \tag{8}$$

We claim that $\lim_{k \rightarrow \infty} d(fx_k, x_{k+2}) = 0$. By (i) and (1) we get

$$\psi(d(fx_k, x_{k+2})) \leq \alpha(fx_{k-1}, x_{k+1})\psi(d((fT)x_{k-1}, Tx_{k+1}))$$

$$\leq \beta(\psi_1(M(fx_{k-1}, x_{k+1})))\psi_2(M(fx_{k-1}, x_{k+1}))$$

$$< \psi_2(M(fx_{k-1}, x_{k+1})), \forall k \in \mathbb{N}$$

(9), where

$$M(fx_{k-1}, x_{k+1}) = \max\{d(fx_{k-1}, x_{k+1}), d(fx_{k-1}, x_k), d(fx_{k+1}, x_{k+2}), d(fx_{k+1}, x_k), d(fx_{k+1}, x_{k+1})\}$$

$$= \max\{d(fx_{k-1}, x_{k+1}), d(fx_{k-1}, x_k)\} \text{ (since } \{d(fx_k, x_{k+1})\} \text{ is decreasing).}$$

(10)

From (9) we get $\psi(d(fx_k, x_{k+2})) < \psi_2(M(fx_{k-1}, x_{k+1})) \leq \psi(M(fx_{k-1}, x_{k+1}))$ (since $\psi_2(t) \leq \psi(t), \forall t > 0$).

Therefore $d(fx_k, x_{k+2}) < \max\{d(fx_{k-1}, x_{k+1}), d(fx_{k-1}, x_k)\}$ (By (10) and since ψ is nondecreasing).
 (11)

From (7) we get $d(fx_k, x_{k+1}) < d(fx_{k-1}, x_k) \leq \max\{d(fx_{k-1}, x_{k+1}), d(fx_{k-1}, x_k)\}$ (By (11))
 (12)

From (11) and (12) we get $\max\{d(fx_k, x_{k+2}), d(fx_k, x_{k+1})\} < \max\{d(fx_{k-1}, x_{k+1}), d(fx_{k-1}, x_k)\}$.
 Therefore the sequence $\{\max\{u_k, v_k\}\}$, where $u_k = d(fx_k, x_{k+2}), v_k = d(fx_k, x_{k+1}), \forall k \in \mathbb{N}$, is a decreasing sequence of nonnegative real numbers so that it converges to some nonnegative real number p .

Therefore $p = \lim_{k \rightarrow \infty} \max\{u_k, v_k\} = \lim_{k \rightarrow \infty} u_k$ (By (8)).
 (13)

i.e., $\lim_{k \rightarrow \infty} d(fx_k, x_{k+2}) = p$
 (14)

Let $p > 0$. Then from (9) we get

$$\begin{aligned} \frac{\psi(d(fx_k, x_{k+2}))}{\psi_2(M(fx_{k-1}, x_{k+1}))} &\leq \beta \left(\psi_1(M(fx_{k-1}, x_{k+1})) \right) < 1 \\ \Rightarrow \frac{\psi(d(fx_k, x_{k+2}))}{\psi(M(fx_{k-1}, x_{k+1}))} &\leq \beta \left(\psi_1(M(fx_{k-1}, x_{k+1})) \right) < 1 \quad (\text{since } \psi_2(t) \leq \psi(t), \forall t > 0). \\ \Rightarrow \frac{\psi(\lim_{k \rightarrow \infty} u_k)}{\psi(\lim_{k \rightarrow \infty} u_k)} &\leq \lim_{k \rightarrow \infty} \beta \left(\psi_1(M(fx_{k-1}, x_{k+1})) \right) \leq 1 \quad (\text{since } \psi \text{ is continuous and by (13)}). \\ \Rightarrow \lim_{k \rightarrow \infty} \beta \left(\psi_1(M(fx_{k-1}, x_{k+1})) \right) &= 1. \\ \Rightarrow \lim_{k \rightarrow \infty} \psi_1(M(fx_{k-1}, x_{k+1})) &= 0 \quad (\text{By property of } \beta) \end{aligned}$$

$\Rightarrow \psi_1(p) = 0$ (By continuity of ψ_1), a contradiction.

$\Rightarrow p = 0$ (By property of ψ_1).

Therefore (14) becomes $\lim_{k \rightarrow \infty} d(fx_k, x_{k+2}) = 0$
 (16)

Let for some $m, k \in \mathbb{N}$ with $m < k, x_k = x_m$. Then $x_{k+1} = Tx_k = Tx_m = x_{m+1}$.

Then (using (7)) $\psi(d(fx_m, x_{m+1})) = \psi(d(fx_k, x_{k+1})) < \psi(d(fx_{k-1}, x_k)) < \dots < \psi(d(fx_m, x_{m+1}))$, a contradiction.

Therefore $x_k \neq x_m, \forall m, k \in \mathbb{N}$ with $m \neq k$. We shall prove that $\{x_k\}$ is I-cauchy. If not, then there exists $\epsilon > 0$ such that

for every $n \in \mathbb{N}, \exists m_n, k_n \in \mathbb{N}$ such that k_n is the smallest positive integer for which

$$k_n \geq m_n > n \text{ and } d(fx_{m_n}, x_{k_n}) \geq \epsilon \quad (17)$$

Then $d(fx_{m_n}, x_{k_n-1}) < \epsilon$
 (18)

Now $\epsilon \leq d(fx_{m_n}, x_{k_n}) \leq d(fx_{m_n}, x_{k_n-1}) + d(fx_{k_n-1}, x_{k_n-2}) + d(fx_{k_n-2}, x_{k_n})$
 $< \epsilon + d(fx_{k_n-2}, x_{k_n-1}) + d(fx_{k_n-2}, x_{k_n})$ (By (17) and (18)).

$$\Rightarrow \lim_{n \rightarrow \infty} d(fx_{m_n}, x_{k_n}) = \epsilon \quad (\text{By (8) and (16)}).$$

(19)

$$\begin{aligned} \text{Again } d(fx_{m_n}, x_{k_n}) &\leq d(fx_{m_n}, x_{k_n+1}) + d(fx_{k_n+1}, x_{k_n+2}) + d(fx_{k_n+2}, x_{k_n}) \\ &\leq d(fx_{m_n}, x_{k_n}) + d(fx_{k_n}, x_{k_n+2}) + d(fx_{k_n+2}, x_{k_n+1}) + d(fx_{k_n+1}, x_{k_n+2}) + d(fx_{k_n+2}, x_{k_n}) \\ &= d(fx_{m_n}, x_{k_n}) + 2d(fx_{k_n}, x_{k_n+2}) + 2d(fx_{k_n+1}, x_{k_n+2}) \\ &\Rightarrow \epsilon \leq \lim_{n \rightarrow \infty} d(fx_{m_n}, x_{k_n+1}) \leq \epsilon \quad (\text{By (8),(16) and (19)}). \end{aligned}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} d(fx_{m_n}, x_{k_n+1}) = \epsilon \tag{20}$$

$$\begin{aligned} \text{Now } d(fx_{m_n}, x_{k_n}) &\leq d(fx_{m_n}, x_{m_n-2}) + d(fx_{m_n-2}, x_{m_n-1}) + d(fx_{m_n-1}, x_{k_n}) \\ &\leq d(fx_{m_n}, x_{m_n-2}) + d(fx_{m_n-2}, x_{m_n-1}) + d(fx_{m_n-1}, x_{m_n-2}) + d(fx_{m_n-2}, x_{m_n}) + d(fx_{m_n}, x_{k_n}) \\ &= d(fx_{m_n}, x_{k_n}) + 2d(fx_{m_n-2}, x_{m_n}) + 2d(fx_{m_n-2}, x_{m_n-1}) \\ &\Rightarrow \epsilon \leq \lim_{n \rightarrow \infty} d(fx_{m_n-1}, x_{k_n}) \leq \epsilon \quad (\text{By (8),(16) and (19)}). \end{aligned}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} d(fx_{m_n-1}, x_{k_n}) = \epsilon \tag{21}$$

$$\begin{aligned} \text{Finally, } d(fx_{m_n-1}, x_{k_n}) &\leq d(fx_{m_n-1}, x_{k_n+1}) + d(fx_{k_n+1}, x_{k_n+2}) + d(fx_{k_n+2}, x_{k_n}) \\ &\leq d(fx_{m_n-1}, x_{k_n}) + d(fx_{k_n}, x_{k_n+2}) + d(fx_{k_n+2}, x_{k_n+1}) + d(fx_{k_n+1}, x_{k_n+2}) + d(fx_{k_n+2}, x_{k_n}) \\ &= d(fx_{m_n-1}, x_{k_n}) + 2d(fx_{k_n}, x_{k_n+2}) + 2d(fx_{k_n+1}, x_{k_n+2}) \\ &\Rightarrow \epsilon \leq \lim_{n \rightarrow \infty} d(fx_{m_n-1}, x_{k_n+1}) \leq \epsilon \quad (\text{By (8),(16) and (21)}). \end{aligned}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} d(fx_{m_n-1}, x_{k_n+1}) = \epsilon$$

(22)

Let $x = x_{m_n-1}, y = x_{k_n}$. Then by (i) we get

$$\begin{aligned} \psi \left(d(fx_{m_n}, x_{k_n+1}) \right) &\leq \alpha(fx_{m_n-1}, x_{k_n}) \psi \left(d \left((fT)x_{m_n-1}, Tx_{k_n} \right) \right) \leq \\ &\beta \left(\psi_1(M(x_{m_n-1}, x_{k_n})) \right) \psi_2(M(x_{m_n-1}, x_{k_n})). \end{aligned} \tag{23}$$

$$\text{Now } \max\{d(fx_{m_n-1}, x_{k_n}), d(fx_{m_n-1}, x_{m_n}), d(fx_{k_n}, x_{k_n+1}), d(fx_{k_n}, x_{m_n}), d(fx_{k_n}, x_{m_n+1})\}. \tag{24}$$

$$\begin{aligned} \text{Now } d(fx_{k_n}, x_{m_n+1}) &\leq d(fx_{k_n}, x_{k_n+1}) + d(fx_{k_n+1}, x_{m_n}) + d(fx_{m_n}, x_{m_n+1}). \\ &\Rightarrow \lim_{n \rightarrow \infty} d(fx_{k_n}, x_{m_n+1}) \leq \epsilon \quad (\text{By (8), (20)}). \end{aligned}$$

$$\begin{aligned} \text{Also, } d(fx_{m_n}, x_{k_n+1}) &\leq d(fx_{m_n}, x_{m_n+1}) + d(fx_{m_n+1}, x_{k_n}) + d(fx_{k_n}, x_{k_n+1}) \\ &\Rightarrow \epsilon \leq \lim_{n \rightarrow \infty} d(fx_{m_n+1}, x_{k_n}) \quad (\text{By (8), (20)}) \end{aligned}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} d(fx_{k_n}, x_{m_n+1}) = \epsilon \tag{25}$$

$$\text{Therefore from (24) we get } \lim_{n \rightarrow \infty} M(x_{m_n-1}, x_{k_n}) = \epsilon \quad (\text{By (8),(21), (19),(25)}).$$

(26)

$$\text{Therefore } \lim_{n \rightarrow \infty} \psi_2(M(x_{m_n-1}, x_{k_n})) = \psi_2(\epsilon) \quad (\text{By continuity of } \psi_2).$$

From (23) we get $\frac{\psi(d(fx_{m_n}, x_{k_n+1}))}{\psi_2(M(x_{m_n-1}, x_{k_n}))} \leq \beta \left(\psi_1 \left(M(x_{m_n-1}, x_{k_n}) \right) \right) < 1$
 $\Rightarrow \frac{\psi(\epsilon)}{\psi_2(\epsilon)} \leq \lim_{n \rightarrow \infty} \beta \left(\psi_1 \left(M(x_{m_n-1}, x_{k_n}) \right) \right) \leq 1$ (By continuity of ψ, ψ_2 and (26)).
 $\Rightarrow 1 = \frac{\psi(\epsilon)}{\psi(\epsilon)} \leq \frac{\psi(\epsilon)}{\psi_2(\epsilon)} \leq \lim_{n \rightarrow \infty} \beta \left(\psi_1 \left(M(x_{m_n-1}, x_{k_n}) \right) \right) \leq 1$ (since $\psi_2(t) \leq \psi(t), \forall t > 0$).
 $\Rightarrow \lim_{n \rightarrow \infty} \beta \left(\psi_1 \left(M(x_{m_n-1}, x_{k_n}) \right) \right) = 1$
 $\Rightarrow \lim_{n \rightarrow \infty} \psi_1 \left(M(x_{m_n-1}, x_{k_n}) \right) = 0$ (By property of β)
 $\Rightarrow \psi_1(\epsilon) = 0$ (By continuity of ψ_1 and (26)), a contradiction.

Therefore $\{x_k\}$ is I-cauchy so that it I-converges to some point $u \in X$ (since X is I-complete).

Then $\lim_{k \rightarrow \infty} d(fx_k, u) = 0$. Therefore $\lim_{k \rightarrow \infty} d(fx_{k+1}, u) = 0$, i. e., $\lim_{k \rightarrow \infty} d((fT)x_k, u) = 0$,

i.e., the sequence $\{Tx_k\}$ I-converges to u . Since T is I-continuous and $\{x_k\}$ I-converges to u , hence $\{Tx_k\}$ I-converges to Tu . Therefore $(fT)u = fu$. Therefore u is an I-fixed point of T in X . Also $\{T^n x_0\} = \{Tx_n\}$ I-converges to u .

Corollary(3.4) Let (X, d) be a complete g.m.s., $\alpha : X^2 \rightarrow \mathbb{R}$ and $T : X \rightarrow X$ satisfying

- (i) T is an J-g α - ψ -Geraghty contraction map such that $\psi_2(t) \leq \psi(t), \forall t > 0$.
- (ii) T is triangular α -admissible.
- (iii) There exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$ and $\alpha(x_0, T^2x_0) \geq 1$.
- (iv) T is continuous.

Then T has a fixed point u in X and $\{T^n x_0\}$ converges to u .

Proof: Replacing the idempotent map f by the identity map I_X on X in Theorem (3.3) we shall get the result.

Now we shall further generalize the generalized \mathcal{L} -contraction in I-g.m.s. and in g.m.s.

Definition(3.5) Let (X, d, f) be an I-g.m.s. and $T : X \rightarrow X$. Then T is said to be an IJg- \mathcal{L} -contraction with respect to $\zeta \in \mathcal{L}$

if there exists $\theta, \phi \in \Omega_{1,2,4}$ and a constant $A \geq 0$ such that for all $x, y \in X$,

$$d((fT)x, Ty) > 0 \Rightarrow \zeta \left(\theta(d((fT)x, Ty)), \phi(d(fx, y) + AN(fx, y)(1 + M(fx, y))) \right) \geq 1, \quad \text{where}$$

$$\phi(t) \leq \theta(t), \forall t > 0,$$

$$\text{and } N(fx, y) = \min \left\{ d(fx, Ty), d(fy, Tx), \frac{d(fx, Tx) + d(fy, Ty)}{2} \right\}, M(fx, y) =$$

$$\max \left\{ d(fx, Ty), d(fy, Tx), \frac{d(fx, Tx) + d(fy, Ty)}{2} \right\}.$$

Replacing the idempotent map f by the identity map I_X on X we shall get a further generalization of generalized

\mathcal{L} -contraction, called Jg- \mathcal{L} -contraction in g.m.s.

“Theorem(3.6) Let (X, d, f) be an I-complete I-g.m.s. and $T : X \rightarrow X$. If T is an I-continuous IJg- \mathcal{L} -contraction map with respect to $\zeta \in \mathcal{L}$, then T has an I-unique I-fixed point in X .

Proof: Let $x_0 \in X$ be arbitrary and $x_n = Tx_{n-1}, \forall n \in \mathbb{N}$. If $fx_n = fx_{n-1}$ for some $n \in \mathbb{N}$, then x_{n-1} is an I-fixed point of T . Let $fx_n \neq fx_{n-1}, \forall n \in \mathbb{N}$. Since T is an IJg- \mathcal{L} -contraction with respect to ζ , and $d((fT)x_{n-1}, Tx_n) = d(fx_n, x_{n+1}) > 0$,

$$\text{we have } 1 \leq \zeta(\theta(d((fT)x_{n-1}, Tx_n)), \phi(d(fx_{n-1}, x_n) + AN(fx_{n-1}, x_n)(1 + M(fx_{n-1}, x_n))))$$

$$< \frac{\phi(d(fx_{n-1}, x_n) + AN(fx_{n-1}, x_n)(1 + M(fx_{n-1}, x_n)))}{\theta(d(fx_n, x_{n+1}))} \quad (\text{By } (\mathcal{L}_2) \text{ of definition of } \mathcal{L}\text{-contraction}).$$

(1)

$$\Rightarrow \theta(d(fx_n, x_{n+1})) < \phi(d(fx_{n-1}, x_n) + AN(fx_{n-1}, x_n)(1 + M(fx_{n-1}, x_n))), \forall n \in \mathbb{N}.$$

(2)

$$\text{Now } N(fx_{n-1}, x_n) = \min \left\{ d(fx_{n-1}, x_{n+1}), d(fx_n, x_n), \frac{d(fx_{n-1}, x_n) + d(fx_n, x_{n+1})}{2} \right\} = 0$$

$$\text{and } M(fx_{n-1}, x_n) = \max \left\{ d(fx_{n-1}, x_{n+1}), d(fx_n, x_n), \frac{d(fx_{n-1}, x_n) + d(fx_n, x_{n+1})}{2} \right\}$$

$$= \max \left\{ d(fx_{n-1}, x_{n+1}), \frac{d(fx_{n-1}, x_n) + d(fx_n, x_{n+1})}{2} \right\}$$

(3)

$$\text{Therefore (2) becomes } \theta(d(fx_n, x_{n+1})) < \phi(d(fx_{n-1}, x_n)) \leq \theta(d(fx_{n-1}, x_n)), \forall n \in \mathbb{N}.$$

(4)

$$\Rightarrow d(fx_n, x_{n+1}) < d(fx_{n-1}, x_n), \forall n \in \mathbb{N} \quad (\text{since } \theta \text{ is nondecreasing})$$

(5)

Therefore $\{d(fx_n, x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers so that it converges to some real

$$\text{number } r \geq 0. \text{ Let } r > 0. \text{ Since } \theta \in \Omega_{1,2,4}, \text{ hence } \lim_{n \rightarrow \infty} \theta(d(fx_n, x_{n+1})) > 1$$

(6)

$$\text{Now } \lim_{n \rightarrow \infty} d(fx_n, x_{n+1}) = \lim_{n \rightarrow \infty} d(fx_{n-1}, x_n) = r \tag{7}$$

$$\text{By continuity of } \theta, \text{ from (6) we get } \theta(r) > 1.$$

(8)

$$\text{Also from (6), (7) and continuity of } \theta, \text{ we get } \lim_{n \rightarrow \infty} \theta(d(fx_{n-1}, x_n)) = \theta(r) > 1$$

(9)

$$\text{Therefore by third property of } \zeta \text{ ((}\mathcal{L}_3\text{) of definition of } \mathcal{L}\text{-contraction) we have}$$

$$1 \leq \overline{\lim}_{n \rightarrow \infty} \zeta(\theta(d(fx_n, x_{n+1})), \theta(d(fx_{n-1}, x_n))) < 1 \tag{10}, \text{ a contradiction.}$$

$$\text{Therefore } \lim_{n \rightarrow \infty} d(fx_n, x_{n+1}) = 0. \tag{11}$$

Let $x_m = x_n$ for some $m, n \in \mathbb{N}$ with $m > n$. Then $Tx_m = Tx_n, i. e., x_{m+1} = x_{n+1}$. Therefore from (4) we get

$$\theta(d(fx_m, x_{m+1})) < \theta(d(fx_{m-1}, x_m)) < \dots < \theta(d(fx_n, x_{n+1})) = \theta(d(fx_m, x_{m+1})), \text{ a contradiction.}$$

Therefore $x_m \neq x_n, \forall m, n \in \mathbb{N}$ with $m \neq n$. Now we shall show that $\{x_n\}$ is I-cauchy. If not, then there exists $\epsilon > 0$ for which there exists two subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ of $\{x_n\}$ such that n_k is the smallest positive integer with

$$n_k > m_k > k \text{ and } d(fx_{m_k}, x_{n_k}) \geq \epsilon. \tag{12}$$

$$\text{Then } d(fx_{m_k}, x_{n_k-i}) < \epsilon \text{ for } i = 1, 2, \dots, n_k \tag{13}$$

$$\begin{aligned} \text{Now } \epsilon \leq d(fx_{m_k}, x_{n_k}) &\leq d(fx_{m_k}, x_{n_k-2}) + d(fx_{n_k-2}, x_{n_k-1}) + d(fx_{n_k-1}, x_{n_k}) \\ &< \epsilon + d(fx_{n_k-2}, x_{n_k-1}) + d(fx_{n_k-1}, x_{n_k}). \end{aligned} \tag{14}$$

Taking limit as $k \rightarrow \infty$ and using (11) we get

$$\begin{aligned} \epsilon &\leq \lim_{k \rightarrow \infty} d(fx_{m_k}, x_{n_k}) \leq \epsilon. \\ \Rightarrow \lim_{k \rightarrow \infty} d(fx_{m_k}, x_{n_k}) &= \epsilon \end{aligned} \tag{15}$$

$$\begin{aligned} \text{Again } d(fx_{m_k}, x_{n_k}) &\leq d(fx_{m_k}, x_{m_k-1}) + d(fx_{m_k-1}, x_{n_k-1}) + d(fx_{n_k-1}, x_{n_k}) \\ &\leq d(fx_{m_k}, x_{m_k-1}) + d(fx_{m_k-1}, x_{m_k}) + d(fx_{m_k}, x_{n_k}) + d(fx_{n_k}, x_{n_k-1}) + \\ &d(fx_{n_k-1}, x_{n_k}) \\ &= 2d(fx_{m_k-1}, x_{m_k}) + 2d(fx_{n_k-1}, x_{n_k}) + d(fx_{m_k}, x_{n_k}) \end{aligned}$$

$$\text{Taking limit as } k \rightarrow \infty \text{ we get } \epsilon \leq \lim_{k \rightarrow \infty} d(fx_{m_k-1}, x_{n_k-1}) \leq \epsilon.$$

$$\Rightarrow \lim_{k \rightarrow \infty} d(fx_{m_k-1}, x_{n_k-1}) = \epsilon \tag{16}$$

Since T is IJg- \mathcal{L} -contraction, we have

$$1 \leq \zeta \left(\theta \left(d \left((fT)x_{m_k-1}, Tx_{n_k-1} \right) \right), \phi \left(d(fx_{m_k-1}, x_{n_k-1}) + B_k \right) \right) < \frac{\phi \left(d(fx_{m_k-1}, x_{n_k-1}) + B_k \right)}{\theta \left(d \left((fT)x_{m_k-1}, Tx_{n_k-1} \right) \right)} \tag{By}$$

property of ζ (17), where

$$B_k = AN(fx_{m_k-1}, x_{n_k-1})(1 + M(fx_{m_k-1}, x_{n_k-1})).$$

$$\text{Now } N(fx_{m_k-1}, x_{n_k-1}) = \min \left\{ d(fx_{m_k-1}, x_{n_k}), d(fx_{n_k-1}, x_{m_k}), \frac{d(fx_{m_k-1}, x_{m_k}) + d(fx_{n_k-1}, x_{n_k})}{2} \right\}, \tag{18}$$

$$M(fx_{m_k-1}, x_{n_k-1}) = \max \left\{ d(fx_{m_k-1}, x_{n_k}), d(fx_{n_k-1}, x_{m_k}), \frac{d(fx_{m_k-1}, x_{m_k}) + d(fx_{n_k-1}, x_{n_k})}{2} \right\} \tag{19}$$

Therefore from (17) we get

$$\begin{aligned} \theta \left(d(fx_{m_k}, x_{n_k}) \right) &< \phi \left(d(fx_{m_k-1}, x_{n_k-1}) + B_k \right), \forall k \in \mathbb{N}. \\ &\leq \theta \left(d(fx_{m_k-1}, x_{n_k-1}) + AN(fx_{m_k-1}, x_{n_k-1}) \left(1 + M(fx_{m_k-1}, x_{n_k-1}) \right) \right), \forall k \in \mathbb{N}. \end{aligned} \tag{20}$$

Since $\lim_{k \rightarrow \infty} N(fx_{m_k-1}, x_{n_k-1}) = 0$, from (20) and by continuity of θ, ϕ we get

$$\lim_{k \rightarrow \infty} \theta \left(d(fx_{m_k}, x_{n_k}) \right) = \lim_{k \rightarrow \infty} \phi \left(d(fx_{m_k-1}, x_{n_k-1}) + B_k \right) = \phi(\epsilon) > 1. \quad (21)$$

From (20), (21), (17) and property of ζ , we get $1 \leq \overline{\lim_{k \rightarrow \infty} \zeta(t_k, s_k)} < 1$, a contradiction, where $t_k = \theta(d(fx_{m_k}, x_{n_k}))$

$$\text{and } s_k = \phi \left(d(fx_{m_k-1}, x_{n_k-1}) + AN(fx_{m_k-1}, x_{n_k-1}) \left(1 + M(fx_{m_k-1}, x_{n_k-1}) \right) \right).$$

Therefore $\{x_n\}$ is I-cauchy in X so that it I-converges to some point z of X (since X is I-complete).

Since T is I-continuous, hence $\{Tx_n\}$ I-converges to Tz . Now $\{Tx_n\} = \{x_{n+1}\}$ I-converges to z .

Therefore $(fT)z = fz$ so that z is an I-fixed point of T .

If possible, let w be an I-fixed point of T in X such that z and w are I-distinct. Then $d(fz, w) > 0$, $(fT)w = fw$.

$$\text{Now } 1 \leq \zeta \left(\theta(d(fz, w)), \phi \left(d(fz, w) + AN(fz, w)(1 + M(fz, w)) \right) \right)$$

$$< \frac{\phi(d(fz, w) + AN(fz, w)(1 + M(fz, w)))}{\theta(d(fz, w))}$$

$$(22), \quad \text{where} \quad N(fz, w) = \min \left\{ d(fz, Tw), d(fw, Tz), \frac{d(fz, Tz) + d(fw, Tw)}{2} \right\} =$$

$$\min \left\{ d(fz, w), d(fw, z), \frac{d(fz, z) + d(fw, w)}{2} \right\} = 0,$$

$$M(fz, w) = \max \left\{ d(fz, Tw), d(fw, Tz), \frac{d(fz, Tz) + d(fw, Tw)}{2} \right\} =$$

$$\max \left\{ d(fz, w), d(fw, z), \frac{d(fz, z) + d(fw, w)}{2} \right\} = d(fz, w).$$

Therefore from (22) we get $\theta(d(fz, w)) < \phi(d(fz, w)) \leq \theta(d(fz, w))$ (since $\phi(t) \leq \theta(t), \forall t > 0$), a contradiction.

Therefore T has an I-unique I-fixed point in X.”

‘Corollary (3.7) Let (X, d) be a complete g.m.s. and $T : X \rightarrow X$. If T is a continuous Jg- \mathcal{L} -contraction map with respect to $\zeta \in \mathcal{L}$, then T has a unique fixed point in X.

Proof: Replacing the idempotent map f by the identity map I_X on X in Theorem(3.6) we shall get the result.”

In Example(3.8) we shall show that Corollary(3.7) is a true extension of Theorem(2.15).

Example(3.8) Let $X = \{1,2,3,4,5\}$ and $d: X^2 \rightarrow [0, \infty)$ is given by $d(1,2) = d(1,4) = 2$, $d(1,5) = d(2,5) = d(3,5) = 1$,

$$d(1,3) = d(2,4) = d(2,3) = d(3,4) = d(4,5) = 4, \quad d(x, x) = 0 \quad \text{for all } x \in X, \quad d(x, y) = d(y, x), \forall x, y \in X.$$

Then (X, d) is a complete g.m.s. but not a metric space, as $4 = d(1,3) > 2 = 1 + 1 = d(1,5) + d(5,3)$.

Let $T: X \rightarrow X$ is given by $Tx = \begin{cases} 3, & x = 1,2,3,4 \\ 2, & x = 5 \end{cases}$

Here T is not a generalized \mathcal{L} -contraction because for $x = 1, y = 5, d(Tx, Ty) = d(3, 2) = 4 > 0$, but for $\zeta \in \mathcal{L}, \theta \in \Omega_{1,2,4}, x = 1, y = 5, A = 2, d(x, y) + AN(x, y) = 1 + 2 = 3 < 4 = d(Tx, Ty)$ so that $\theta(d(x, y) + AN(x, y)) < \theta(d(Tx, Ty))$, since θ is nondecreasing. This contradicts the property (\mathcal{L}_2) for ζ , in this case. Therefore Theorem(2.15) cannot be used here.

Now, $d(Tx, Ty) = \begin{cases} 4, & \text{if } (x = 5, y \neq 5) \text{ or } (x \neq 5, y = 5) \\ 0, & \text{otherwise} \end{cases}$

Consider $\zeta \in \mathcal{L}$ given by $\zeta(t, s) = \frac{\sqrt{s}}{t}, \forall t, s \geq 1, A = 2$. Take any $\theta, \phi \in \Omega_{1,2,4}$ with $\phi(t) = \theta(t), \forall t > 0$. Now $\forall x, y \in X$ with $d(Tx, Ty) > 0$, it can be easily verified that

$$\zeta\left(\theta(d(Tx, Ty)), \phi\left(d(x, y) + AN(x, y)(1 + M(x, y))\right)\right) = \frac{\sqrt{\theta\left(d(x, y) + 2N(x, y)(1 + M(x, y))\right)}}{\theta(d(Tx, Ty))} \geq 1$$

Therefore by Corollary(3.7), T has a unique fixed point in X and $x = 3$ is that fixed point.

Note(3.9) Now on the basis of the concept of IJg- \mathcal{L} -contraction, we shall introduce a new contraction of a pair of self-map

of an I-g.m.s. and establish a common fixed point result of this pair of maps in I-g.m.s. Then we shall get the corresponding version of this common fixed point result in g.m.s. by means of a corollary.

Definition(3.10) Let (X, d, f) be an I-g.m.s. and $S, T : X \rightarrow X$. Then S and T are said to be cooperatively IJ- \mathcal{L} -contraction with respect to $\zeta \in \mathcal{L}$ if there exists $\theta \in \Omega_{1,2,4}$ and a constant $A \geq 0$ such that for all $x, y \in X$,

$$d((fS)x, Ty) > 0 \implies \zeta\left(\theta(d((fS)x, Ty)), \theta(d(fx, y) + AN(fx, y)(1 + M(fx, y)))\right) \geq 1, \text{ where}$$

$$N(fx, y) = \min\left\{d(fx, Ty), d(fy, Sx), \frac{d(fx, Sx) + d(fy, Ty)}{2}\right\}, \quad M(fx, y) = \max\left\{d(fx, Ty), d(fy, Sx), \frac{d(fx, Sx) + d(fy, Ty)}{2}\right\}.$$

Replacing the idempotent map f by the identity map I_X on X we shall get the corresponding version of this contraction in g.m.s., called cooperatively J- \mathcal{L} -contraction with respect to $\zeta \in \mathcal{L}$.

Theorem(3.11) Let (X, d, f) be an I-complete I-g.m.s. and $S, T : X \rightarrow X$. If S, T are cooperatively IJ- \mathcal{L} -contraction with respect to $\zeta \in \mathcal{L}$, then S and T have an I-unique common I-fixed point in X .

Proof: Firstly we shall show that if u be an I-fixed point of one of S and T , then u is an I-fixed point of the other. Let u be

an I-fixed point of S so that $(fS)u = fu$. If possible, let u is not an I-fixed point of T . Then $d(fu, Tu) = d((fS)u, Tu) > 0$. This implies that $1 \leq \zeta\left(\theta(d(fu, Tu)), \theta\left(d(fu, u) + AN(fu, u)(1 + M(fu, u))\right)\right)$

$$= \zeta(\theta(d(fu, Tu)), \theta(0)), \text{ since } N(fu, u) = \min \left\{ d(fu, Tu), d(fu, Su), \frac{d(fu, Su) + d(fu, Tu)}{2} \right\} = 0.$$

Therefore $1 \leq \zeta(\theta(d(fu, Tu)), \theta(0)) < \frac{\theta(0)}{\theta(d(fu, Tu))} \leq 1$, a contradiction. Therefore u is an I-fixed point of T .

Similarly, we can show that if u be an I-fixed point of T , then u is an I-fixed point of S .

Let $x_0 \in X$ be arbitrary and $x_1 = Tx_0, x_{2k} = Sx_{2k-1}, x_{2k+1} = Tx_{2k} \forall k \in \mathbb{N}$. If $fx_{2k} = fx_{2k-1}$ for some $k \in \mathbb{N}$, then x_{2k-1}

is an I-fixed point of S and hence x_{2k-1} is an I-fixed point of T so that x_{2k-1} is a common I-fixed point of S and T .

If $fx_{2k} = fx_{2k+1}$ for some $k \geq 0$, then x_{2k} is an I-fixed point of T and hence x_{2k} is an I-fixed point of S so that x_{2k} is a common I-fixed point of S and T .

Let $fx_n \neq fx_{n-1}, \forall n \in \mathbb{N}$. Let $u_k = d(fx_k, x_{k+1}), \forall k \in \mathbb{N} \cup \{0\}$. Then $u_k > 0, \forall k \in \mathbb{N} \cup \{0\}$.

For $k = 2q (q \geq 0)$, since S and T are cooperatively IJ- \mathcal{L} -contraction, and since

$u_k = d(fx_k, x_{k+1}) = d((fS)x_{k-1}, Tx_k) > 0$, we have

$$\zeta \left(\theta \left(d((fS)x_{k-1}, Tx_k) \right), \theta \left(d(fx_{k-1}, x_k) + AN((fS)x_{k-1}, Tx_k)(1 + M((fS)x_{k-1}, Tx_k)) \right) \right) \geq 1,$$

$$\text{where } N((fS)x_{k-1}, Tx_k) = \min \left\{ d(fx_{k-1}, Tx_k), d(fx_k, Sx_{k-1}), \frac{d(fx_{k-1}, Sx_{k-1}) + d(fx_k, Tx_k)}{2} \right\}$$

$$\text{and } M((fS)x_{k-1}, Tx_k) = \max \left\{ d(fx_{k-1}, Tx_k), d(fx_k, Sx_{k-1}), \frac{d(fx_{k-1}, Sx_{k-1}) + d(fx_k, Tx_k)}{2} \right\}.$$

$$\Rightarrow \zeta \left(\theta \left(d(fx_k, x_{k+1}) \right), \theta \left(d(fx_{k-1}, x_k) + AN(fx_k, x_{k+1})(1 + M(fx_k, x_{k+1})) \right) \right) \geq 1,$$

$$\text{where } N(fx_k, x_{k+1}) = \min \left\{ d(fx_{k-1}, x_{k+1}), d(fx_k, x_k), \frac{d(fx_{k-1}, x_k) + d(fx_k, x_{k+1})}{2} \right\} = 0$$

$$\text{and } M((fS)x_{k-1}, Tx_k) = \max \left\{ d(fx_{k-1}, x_{k+1}), d(fx_k, x_k), \frac{d(fx_{k-1}, x_k) + d(fx_k, x_{k+1})}{2} \right\}$$

$$= \max \left\{ d(fx_{k-1}, x_{k+1}), \frac{d(fx_{k-1}, x_k) + d(fx_k, x_{k+1})}{2} \right\}.$$

$$\Rightarrow 1 \leq \zeta \left(\theta \left(d(fx_k, x_{k+1}) \right), \theta \left(d(fx_{k-1}, x_k) \right) \right) < \frac{\theta(d(fx_{k-1}, x_k))}{\theta(d(fx_k, x_{k+1}))} \quad (\text{By property of } \zeta \text{ and since } \theta \in \Omega_{1,2,4}).$$

(1)

$$\Rightarrow \theta(d(fx_k, x_{k+1})) < \theta(d(fx_{k-1}, x_k)), \forall k \in \mathbb{N}.$$

(2)

$$\Rightarrow d(fx_k, x_{k+1}) < d(fx_{k-1}, x_k), \forall k \in \mathbb{N}. \quad (\text{since } \theta \text{ is non- decreasing}).$$

(3)

Therefore $\{d(fx_k, x_{k+1})\}$ is a decreasing sequence of nonnegative real numbers, so that it converges to some real

number $r \geq 0$. Let $r > 0$. Since $\theta \in \Omega_{1,2,4}$, hence $\lim_{k \rightarrow \infty} \theta(d(fx_k, x_{k+1})) > 1$.

(4)

Now, $\lim_{k \rightarrow \infty} \theta(d(fx_k, x_{k+1})) = \lim_{k \rightarrow \infty} \theta(d(fx_{k-1}, x_k)) = \theta(r) > 1$ (By (4), by continuity of θ)

(5)

From (1), (2), (5) and third property of ζ (\mathcal{L}_3) of definition of \mathcal{L} -contraction, we have

$$1 \leq \overline{\lim}_{k \rightarrow \infty} \zeta \left(\theta(d(fx_k, x_{k+1})), \theta(d(fx_{k-1}, x_k)) \right) < 1 \tag{6}, \quad a$$

contradiction.

Therefore $r = 0$ so that $\lim_{k \rightarrow \infty} d(fx_k, x_{k+1}) = 0$.

(7)

For $k = 2q + 1$ ($q \geq 0$), since S and T are cooperatively IJ - \mathcal{L} -contraction, and since $u_k = d(fx_k, x_{k+1}) = d((fS)x_k, Tx_{k-1}) > 0$, we have

$$\zeta \left(\theta \left(d((fS)x_k, Tx_{k-1}) \right), \theta(d(fx_k, x_{k-1}) + AN((fS)x_k, Tx_{k-1})(1 + M((fS)x_k, Tx_{k-1}))) \right) \geq 1,$$

where $N((fS)x_k, Tx_{k-1}) = \min \left\{ d(fx_k, Tx_{k-1}), d(fx_{k-1}, Sx_k), \frac{d(fx_k, Sx_k) + d(fx_{k-1}, Tx_{k-1})}{2} \right\}$

and $M((fS)x_k, Tx_{k-1}) = \max \left\{ d(fx_k, Tx_{k-1}), d(fx_{k-1}, Sx_k), \frac{d(fx_k, Sx_k) + d(fx_{k-1}, Tx_{k-1})}{2} \right\}$

$$\Rightarrow \zeta \left(\theta(d(fx_{k+1}, x_k)), \theta(d(fx_k, x_{k-1}) + AN(fx_{k+1}, x_k)(1 + M(fx_{k+1}, x_k))) \right) \geq 1,$$

where $N(fx_{k+1}, x_k) = \min \left\{ d(fx_k, x_k), d(fx_{k-1}, x_{k+1}), \frac{d(fx_k, x_{k+1}) + d(fx_{k-1}, x_k)}{2} \right\} = 0$

and $M(fx_{k+1}, x_k) = \max \left\{ d(fx_k, x_k), d(fx_{k-1}, x_{k+1}), \frac{d(fx_k, x_{k+1}) + d(fx_{k-1}, x_k)}{2} \right\}$

$$= \max \left\{ d(fx_{k-1}, x_{k+1}), \frac{d(fx_k, x_{k+1}) + d(fx_{k-1}, x_k)}{2} \right\}.$$

$$\Rightarrow 1 \leq \zeta \left(\theta(d(fx_k, x_{k+1})), \theta(d(fx_{k-1}, x_k)) \right) < \frac{\theta(d(fx_{k-1}, x_k))}{\theta(d(fx_k, x_{k+1}))} \quad (\text{By property of } \zeta \text{ and since } \theta \in \Omega_{1,2,4}).$$

(8)

$$\Rightarrow \theta(d(fx_k, x_{k+1})) < \theta(d(fx_{k-1}, x_k)), \forall k \in \mathbb{N}.$$

(9)

$$\Rightarrow d(fx_k, x_{k+1}) < d(fx_{k-1}, x_k), \forall k \in \mathbb{N}. \quad (\text{since } \theta \text{ is nondecreasing}).$$

(10)

Therefore $\{d(fx_k, x_{k+1})\}$ is a decreasing sequence of nonnegative real numbers, so that it converges to some

real number $r \geq 0$. Let $r > 0$. Since $\theta \in \Omega_{1,2,4}$, hence $\lim_{k \rightarrow \infty} \theta(d(fx_k, x_{k+1})) > 1$.

(11)

Now, $\lim_{k \rightarrow \infty} \theta(d(fx_k, x_{k+1})) = \lim_{k \rightarrow \infty} \theta(d(fx_{k-1}, x_k)) = \theta(r) > 1$ (By (11), by continuity of θ).

(12)

From (8), (9), (12) and third property of ζ ((\mathcal{L}_3) of definition of \mathcal{L} -contraction), we have

$$1 \leq \overline{\lim}_{k \rightarrow \infty} \zeta \left(\theta(d(fx_k, x_{k+1})), \theta(d(fx_{k-1}, x_k)) \right) < 1 \tag{13}, a$$

contradiction.

Therefore $r = 0$ so that $\lim_{k \rightarrow \infty} d(fx_k, x_{k+1}) = 0$.

(14)

Therefore, in any case, from (7) and (14) we have $\lim_{k \rightarrow \infty} d(fx_k, x_{k+1}) = 0$.

(15)

Now we shall show that $\{x_k\}$ is I-cauchy. If not, then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists two positive integers $2p_n + 1, 2q_n$ with $2p_n + 1$ is the smallest positive integer for which $2p_n + 1 > 2q_n > n$ and $d(fx_{2q_n}, x_{2p_n+1}) \geq \epsilon$.

$$(16)$$

Then $d(fx_{2q_n}, x_{2p_n+1-2i}) < \epsilon$ for $i = 1, 2, \dots, p_n$.

$$(17)$$

Now $\epsilon \leq d(fx_{2q_n}, x_{2p_n+1}) \leq d(fx_{2q_n}, x_{2p_n-1}) + d(fx_{2p_n-1}, x_{2p_n}) + d(fx_{2p_n}, x_{2p_n+1}) < \epsilon + d(fx_{2p_n-1}, x_{2p_n}) + d(fx_{2p_n}, x_{2p_n+1})$ (By (17)).

$$(18)$$

Taking limit as $n \rightarrow \infty$ in (18) we get

$$\epsilon \leq \lim_{k \rightarrow \infty} d(fx_{2q_n}, x_{2p_n+1}) \leq \epsilon. \text{ (By (15)).}$$

$$\Rightarrow \lim_{k \rightarrow \infty} d(fx_{2q_n}, x_{2p_n+1}) = \epsilon.$$

$$(19)$$

Again $d(fx_{2q_n}, x_{2p_n+1}) \leq d(fx_{2q_n}, x_{2q_n-1}) + d(fx_{2q_n-1}, x_{2p_n}) + d(fx_{2p_n}, x_{2p_n+1}) \leq d(fx_{2q_n}, x_{2q_n-1}) + d(fx_{2q_n-1}, x_{2q_n}) + d(fx_{2q_n}, x_{2p_n+1}) + d(fx_{2p_n+1}, x_{2p_n}) + d(fx_{2p_n}, x_{2p_n+1}) = 2d(fx_{2q_n-1}, x_{2q_n}) + d(fx_{2q_n}, x_{2p_n+1}) + 2d(fx_{2p_n}, x_{2p_n+1})$

Taking limit as $n \rightarrow \infty$ we get $\epsilon \leq \lim_{k \rightarrow \infty} d(fx_{2q_n-1}, x_{2p_n}) \leq \epsilon$ (By (15)).

$$\Rightarrow \lim_{k \rightarrow \infty} d(fx_{2q_n-1}, x_{2p_n}) = \epsilon$$

$$(20)$$

Since S and T are cooperatively IJ- \mathcal{L} -contraction with respect to ζ , and since

$d(fx_{2q_n}, x_{2p_n+1}) = d((fS)x_{2q_n-1}, Tx_{2p_n}) > 0$, we have

$$\zeta \left(\theta \left(d(fx_{2q_n}, x_{2p_n+1}) \right), \theta \left(d(fx_{2q_n-1}, x_{2p_n}) + AN(fx_{2q_n}, x_{2p_n+1}) \left(1 + M(fx_{2q_n}, x_{2p_n+1}) \right) \right) \right) \geq 1,$$

where $N(fx_{2q_n}, x_{2p_n+1}) = \min \left\{ d(fx_{2q_n-1}, x_{2p_n+1}), d(fx_{2p_n}, x_{2q_n}), \frac{d(fx_{2q_n-1}, x_{2q_n}) + d(fx_{2p_n}, x_{2p_n+1})}{2} \right\}$

and $M(fx_{2q_n}, x_{2p_n+1}) = \max \left\{ d(fx_{2q_n-1}, x_{2p_n+1}), d(fx_{2p_n}, x_{2q_n}), \frac{d(fx_{2q_n-1}, x_{2q_n}) + d(fx_{2p_n}, x_{2p_n+1})}{2} \right\}$

$$(21)$$

$$\Rightarrow 1 \leq \zeta \left(\theta \left(d(fx_{2q_n}, x_{2p_n+1}) \right), \theta \left(d(fx_{2q_n-1}, x_{2p_n}) + AN(fx_{2q_n}, x_{2p_n+1}) \left(1 + \right. \right. \right.$$

$$\left. \left. M(fx_{2q_n}, x_{2p_n+1}) \right) \right) \right)$$

$$< \frac{\theta \left(d(fx_{2q_n-1}, x_{2p_n}) + AN(fx_{2q_n}, x_{2p_n+1}) \left(1 + M(fx_{2q_n}, x_{2p_n+1}) \right) \right)}{\theta \left(d(fx_{2q_n}, x_{2p_n+1}) \right)}$$

$$\Rightarrow \theta \left(d(fx_{2q_n}, x_{2p_n+1}) \right) < \theta \left(d(fx_{2q_n-1}, x_{2p_n}) + AN(fx_{2q_n}, x_{2p_n+1}) \left(1 + M(fx_{2q_n}, x_{2p_n+1}) \right) \right)$$

(22)

Let $t_n = \theta \left(d(fx_{2q_n}, x_{2p_n+1}) \right)$, $s_n = \theta \left(d(fx_{2q_n-1}, x_{2p_n}) + AN(fx_{2q_n}, x_{2p_n+1}) \left(1 + M(fx_{2q_n}, x_{2p_n+1}) \right) \right)$.

Then by (22) we have $t_n < s_n, \forall n \in \mathbb{N}$.

Again taking limit as $n \rightarrow \infty$ in (22) and using (15), (19), (20), and continuity of θ we get

$$\lim_{n \rightarrow \infty} t_n = \theta(\epsilon) = \lim_{n \rightarrow \infty} s_n, \text{ since } \lim_{n \rightarrow \infty} d(fx_{2q_n-1}, x_{2q_n}) = 0 = \lim_{n \rightarrow \infty} d(fx_{2p_n}, x_{2p_n+1}) \text{ (By (15))}$$

This implies that $\lim_{n \rightarrow \infty} N(fx_{2q_n}, x_{2p_n+1}) = 0$.

Therefore $\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} s_n = \theta(\epsilon) > 1$, since

$$0 < \epsilon = \lim_{n \rightarrow \infty} d(fx_{2q_n}, x_{2p_n+1}) = \lim_{n \rightarrow \infty} d(fx_{2q_n-1}, x_{2p_n}) \text{ (By (19), (20) and property of } \theta \text{)}.$$

Therefore $1 \leq \overline{\lim}_{n \rightarrow \infty} \zeta(t_n, s_n) < 1$, a contradiction.

Therefore $\{x_k\}$ is I-cauchy in X so that it I-converges to some point u of X (since X is I-complete).

We shall show that u is a common I-fixed point of S and T .

Now $d(fx_k, Tu) \leq d(fx_k, x_{k+1}) + d(fx_{k+1}, u) + d(fu, Tu)$

$$\Rightarrow \lim_{k \rightarrow \infty} d(fx_k, Tu) \leq d(fu, Tu) \text{ (By (15) and } \lim_{k \rightarrow \infty} x_k = u \text{)}.$$

(23)

Again $d(fu, Tu) \leq d(fu, x_{k-1}) + d(fx_{k-1}, x_k) + d(fx_k, Tu)$

$$\Rightarrow d(fu, Tu) \leq \lim_{k \rightarrow \infty} d(fx_k, Tu) \text{ (By (15) and } \lim_{k \rightarrow \infty} x_k = u \text{)}.$$

(24)

From (23) and (24) we get $\lim_{k \rightarrow \infty} d(fx_k, Tu) = d(fu, Tu)$

(25)

We claim that $d(fu, Tu) = 0$. Let $d(fu, Tu) > 0$.

Then from (25) we have $\lim_{k \rightarrow \infty} \theta(d(fx_k, Tu)) > 1$, i. e., $\theta(d(fu, Tu)) > 1$.

Since S and T are cooperatively IJ- \mathcal{L} -contraction, and since $d(fx_{2k}, Tu) = d((fS)x_{2k-1}, Tu) > 0$,

$\zeta \left(\theta(d(fx_{2k}, Tu)), \theta(d(fx_{2k-1}, u) + AN(fx_{2k}, Tu)(1 + M(fx_{2k}, Tu))) \right) \geq 1$, where

$$N(fx_{2k}, Tu) = \min \left\{ d(fx_{2k-1}, Tu), d(fu, x_{2k}), \frac{d(fx_{2k-1}, x_{2k}) + d(fu, Tu)}{2} \right\} \text{ and}$$

$$M(fx_{2k}, Tu) = \max \left\{ d(fx_{2k-1}, Tu), d(fu, x_{2k}), \frac{d(fx_{2k-1}, x_{2k}) + d(fu, Tu)}{2} \right\}$$

(26)

Now $1 \leq \zeta \left(\theta(d(fx_{2k}, Tu)), \theta(d(fx_{2k-1}, u) + AN(fx_{2k}, Tu)(1 + M(fx_{2k}, Tu))) \right)$

$$< \frac{\theta(d(fx_{2k-1}, u) + AN(fx_{2k}, Tu)(1 + M(fx_{2k}, Tu)))}{\theta(d(fx_{2k}, Tu))} \text{ (By property of } \zeta \text{)}$$

$$\Rightarrow \theta(d(fx_{2k}, Tu)) < \theta(d(fx_{2k-1}, u) + AN(fx_{2k}, Tu)(1 + M(fx_{2k}, Tu)))$$

(27)

Taking limit as $k \rightarrow \infty$ in (27), using (15), (25), $\lim_{k \rightarrow \infty} x_k = u$ and continuity of θ we get $1 < \theta(d(fu, Tu)) \leq 1$, a contradiction, since $\lim_{k \rightarrow \infty} d(fu, x_{2k}) = 0 \Rightarrow \lim_{k \rightarrow \infty} N(fx_{2k}, Tu) = 0$.

Therefore $d(fu, Tu) = 0$ so that $(fT)u = fu$.

Therefore u is an I-fixed point of T in X . Hence u is an I-fixed point of S in X also. Therefore u is a common I-fixed point of S and T in X .

Let v be a common I-fixed point of S and T such that $fv \neq fu$.

Then $d(fu, v) > 0$, $(fS)v = fv = (fT)v$, $\theta(d(fu, v)) > 1$.

Since S and T are cooperatively IJ- \mathcal{L} -contraction and $d(fu, v) = d((fS)u, Tv) > 0$, we have

$$\zeta(\theta(d(fu, v)), \theta(d(fu, v) + AN(fu, v)(1 + M(fu, v)))) \geq 1$$

$$\Rightarrow 1 \leq \zeta\left(\theta(d(fu, v)), \theta\left(d(fu, v) + AN(fu, v)(1 + M(fu, v))\right)\right)$$

$$< \frac{\theta\left(d(fu, v) + AN(fu, v)(1 + M(fu, v))\right)}{\theta(d(fu, v))}$$

$\Rightarrow \theta(d(fu, v)) < \theta\left(d(fu, v) + AN(fu, v)(1 + M(fu, v))\right)$, where

$$N(fu, v) = \min\left\{d(fu, v), d(fv, u), \frac{d(fu, u) + d(fv, v)}{2}\right\} = 0,$$

$$M(fu, v) = \max\left\{d(fu, v), d(fv, u), \frac{d(fu, u) + d(fv, v)}{2}\right\} = d(fu, v).$$

(28)

$\Rightarrow 1 < \theta(d(fu, v)) < \theta(d(fu, v))$, a contradiction.

Therefore $fu = fv$. Therefore S and T have I-unique common I-fixed point in X .

Corollary(3.12) Let (X, d) be a complete g.m.s. and $S, T : X \rightarrow X$. If S, T are cooperatively J- \mathcal{L} -contraction with respect

to $\zeta \in \mathcal{L}$, then S and T have a unique common fixed point in X .

Proof: Replacing the idempotent map f by the identity map I_X on X in Theorem(3.10) we shall get the result.

Conclusion

Further study may be continued to develop more results.

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